TEST
Diffusion and Dispersion in ADH

Gary L. Brown

Diffusion and Dispersion Processes

The eddy viscosity terms in the ADH 2-D hydrodynamics, and the turbulent diffusion terms in the ADH 2-D transport, are given by a combination of 3 separate processes, lumped into one term. The processes are:

Turbulent diffusion (TDF) – the process of mixing due to the exchange of fluid parcels at the scale of the turbulence in the flow

Streamwise Dispersion (SDP) – the "apparent" diffusion due to the streamwise dispersion of mass induced by the vertical variation of the velocity profile.

Transverse Dispersion (TDP) - the "apparent" diffusion due to the transverse dispersion of mass induced by transverse circulation in bendways (this only occurs when vorticity transport is active).

A development of the expressions used to quantify each of these processes is given below:

Turbulent diffusion (TDF)

The transverse turbulent diffusion was investigated experimentally by Webel and Schatzmann (1984). They found that, for most natural systems, the width-to-depth ratio is sufficient to render side-wall effects negligible with respect to the transverse mixing, and that the transverse turbulent diffusion can be well represented with the following equation:

$$\varepsilon_{TDF} \cong 0.13 h u_*$$

For ADH, we take this value as the isotropic turbulent diffusion.

This relationship can be given as a function of the depth-averaged velocity using the following equation:

$$\frac{\overline{u}}{u_*} = \sqrt{\frac{2}{C_d}}$$

Hence, the final equation for the isotropic turbulent diffusion is given as follows:

$$\boxed{\varepsilon_{TDF} = 0.092\alpha\sqrt{C_d}\, h\overline{u}} \quad \boxed{D_{TDF} = 0.092\alpha\sqrt{C_d}\, h\overline{u}}$$

Where α is a user defined adjustment factor (default = 1.0)
In order to ensure that some minimum value of the turbulent diffusion is always applied, a minimum velocity is specified for the turbulent diffusion equation. This minimum velocity is calculated assuming that the local Froude number is equal to 0.1.

$$\boxed{\bar{u}_{MIN} = 0.1\sqrt{gh}}$$

Streamwise Dispersion (SDP)

Streamwise dispersion is only relevant when depth averaging. It is a measure of the streamwise spreading of the fluid mass due to the shape of the velocity profile. It is given as follows:

$$\varepsilon_{SDP} = 2\int_{\frac{1}{e}h}^{h} \left(u(z) - \bar{u}\right) dz$$

$$\bar{u} = \frac{1}{h}\int_0^h u(z)\, dz$$

The value h/e is given because this represents the depth at which the velocity is equal to the depth averaged velocity.

The standard logarithmic velocity profile is given as follows:

$$\frac{u}{u_*} = \frac{1}{\kappa}\ln\left(\frac{29.7z}{k}\right)$$

Using this profile, we find the following expression for streamwise dispersion:

$$\varepsilon_{SDP} = \frac{2}{e}\frac{h}{\kappa}\sqrt{\frac{C_d}{2}}\,\bar{u}$$

$$\boxed{\varepsilon_{SDP} = 1.3\alpha\sqrt{C_d}\,h\bar{u}} \quad \boxed{D_{SDP} = 1.3\alpha\sqrt{C_d}\,h\bar{u}}$$

Transverse Dispersion (TDP)

Transverse dispersion is only relevant when vorticity transport is active. It is a measure of the transverse spreading of the fluid mass due to the helical flow in bendways. It is given as follows:

$$\varepsilon_{TDP}=2\int_{\frac{1}{2}h}^{h}\left(u_T\left(z\right)-\overline{u_T}\right)dz$$

$$u_T = u_{T.MAX}\left(2\frac{z}{h}-1\right)$$

$$\overline{u}_T=0$$

Using these equations, we find the following expression for transverse dispersion:

$$\boxed{\varepsilon_{TDP} = 0.5\alpha h u_{T.MAX}} \quad \boxed{D_{TDP} = 0.5\alpha h u_{T.MAX}}$$

Equations for Eddy Viscosity and Turbulent Diffusion in ADH

The equations for isotropic eddy viscosity and turbulent diffusion are given below:

$$EV_{XX}=2\varepsilon_{XX}\frac{\partial u}{\partial x}$$

$$EV_{YY}=2\varepsilon_{YY}\frac{\partial v}{\partial y}$$

$$EV_{XY}=\varepsilon_{XY}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$EV_{YX}=\varepsilon_{YX}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$DF_{XX}=D_{XX}\frac{\partial u}{\partial x}$$

$$DF_{YY}=D_{YY}\frac{\partial u}{\partial x}$$

In ADH, the eddy viscosity and turbulent diffusion terms have an isotropic component, a streamwise component, and a transverse component

$$\varepsilon=\varepsilon_{TDF}+\varepsilon_{SDP}+\varepsilon_{TDP}$$

$$D=D_{TDF}+D_{SDP}+D_{TDP}$$

The directional terms (streamwise and transverse) are given by calculating a coordinate transformation of the full turbulent mixing term. So, for example:

$$EV_{\eta\eta}=2\varepsilon_{\eta\eta}\frac{\partial u}{\partial \eta} = 2\varepsilon_{\eta\eta}\sqrt{\left(\alpha\left(\alpha\frac{\partial u}{\partial x}+\beta\frac{\partial u}{\partial y}\right)\right)^2+\left(\beta\left(\beta\frac{\partial u}{\partial y}+\alpha\frac{\partial u}{\partial x}\right)\right)^2}$$

Where α and β are the directional cosines of the streamwise velocity (α =u/U, β =v/U)

The directional cosines of the transverse velocity are given as follows:

$\alpha_T = -\beta$

$\beta_T = \alpha$

Hence the full turbulence terms, with the isotropic component, the streamwise component, and the transverse component included, are given as follows:

$$\boxed{EV_{XX} = 2\varepsilon_{SDP}\alpha\left(\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y}\right) + 2\varepsilon_{TDP}\alpha_T\left(\alpha_T\frac{\partial u}{\partial x} + \beta_T\frac{\partial u}{\partial y}\right) + 2\varepsilon_{TDF}\frac{\partial u}{\partial x}}$$

$$\boxed{EV_{YY} = 2\varepsilon_{SDP}\beta\left(\beta\frac{\partial v}{\partial y} + \alpha\frac{\partial v}{\partial x}\right) + 2\varepsilon_{TDP}\beta_T\left(\beta_T\frac{\partial v}{\partial y} + \alpha_T\frac{\partial v}{\partial x}\right) + 2\varepsilon_{TDF}\frac{\partial v}{\partial y}}$$

$$\boxed{EV_{XY} = \varepsilon_{SDP}\beta\left(\beta\frac{\partial u}{\partial y} + \alpha\frac{\partial u}{\partial x} + \alpha\frac{\partial v}{\partial x} + \beta\frac{\partial v}{\partial y}\right) + \varepsilon_{TDP}\beta_T\left(\beta_T\frac{\partial u}{\partial y} + \alpha_T\frac{\partial u}{\partial x} + \alpha_T\frac{\partial v}{\partial x} + \beta_T\frac{\partial v}{\partial y}\right) + \varepsilon_{TDF}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)}$$

$$\boxed{EV_{YX} = \varepsilon_{SDP}\alpha\left(\beta\frac{\partial u}{\partial y} + \alpha\frac{\partial u}{\partial x} + \alpha\frac{\partial v}{\partial x} + \beta\frac{\partial v}{\partial y}\right) + \varepsilon_{TDP}\alpha_T\left(\beta_T\frac{\partial u}{\partial y} + \alpha_T\frac{\partial u}{\partial x} + \alpha_T\frac{\partial v}{\partial x} + \beta_T\frac{\partial v}{\partial y}\right) + \varepsilon_{TDF}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)}$$

$$\boxed{DF_{XX} = D_{SDP}\alpha\left(\alpha\frac{\partial C}{\partial x} + \beta\frac{\partial C}{\partial y}\right) + D_{TDP}\alpha_T\left(\alpha_T\frac{\partial C}{\partial x} + \beta_T\frac{\partial C}{\partial y}\right) + D_{TDF}\frac{\partial C}{\partial x}}$$

$$\boxed{DF_{YY} = D_{SDP}\beta\left(\beta\frac{\partial C}{\partial y} + \alpha\frac{\partial C}{\partial x}\right) + D_{TDP}\beta_T\left(\beta_T\frac{\partial C}{\partial y} + \alpha_T\frac{\partial C}{\partial x}\right) + D_{TDF}\frac{\partial C}{\partial y}}$$

Reference

Webel, G. and Schatzmann, M. (1984). “Transverse Mixing in Open Channel Flow” *Journal of Hydraulic Engineering*, Vol 110, No. 4, April 1984, pp 423-435.